3P-9

Proceedings of Symposium on Ultrasonic Electronics, Vol. 31 (2010) pp. 473-474
6-8 December, 2010

Application of Complex Series Dynamics in Some Boundary Conditions

複素級数力学の幾つかの境界条件への適用

Michio Ohki (National Defense Academy, Japan)
大木 道生 (防大)

1. Introduction

The characteristics of electromechanical coupling phenomenon can be represented using lumped parameters, which is useful when the (global) characteristics in the transducer as a whole are observed. From the local point of view, however, the characteristics should be expressed with distributed parameters. The previous equivalent circuit models include some lumped parameters: For example, Mason's circuit includes a pair of positive and negative capacitance, $\pm C_0$, but the stored energy in $-C_0$ alone is meaningless. From the viewpoint of considering energy and its local property, another method is desired. The author has developed the method of the vibration analysis on the distributed-parameter basis, by considering the flow of energy and its superposition, and applied to some problems.[1-3] Such process is expressed by multiplying some kinds of matrices with complex (non real number) elements and by considering the infinite geometric series of matrices, Neumann series, mathematically. This method is termed "complex series dynamics".

In this study, the treatment of the above-mentioned coupling phenomenon is developed along our methodology in the case of multiple-layered system using matrix operation: The process of energy exchange between an elastic mode denoted by η_c and a dielectric mode denoted by η_d is considered, in which η_c and η_d are expressed as *N*-by-1 matrices, respectively, when *N*-layered system is considered. The coupling phenomenon is expressed with a *2N*-by-*2N* unitary matrix mathematically or energy-conservative process physically. Two types of unitary processes are considered in order to treat the elastic-dielectric energy coupling. One process occurs at the boundary or edges of the layer inside the transducer in a spatially discrete manner, and the other process occurs along the layer in a spatially continuous manner. These two types of interaction should be distinguished in the theory (let us call the former "point interaction" and the latter "continuous interaction"), when there exist multiple layers inside the transducer. (In the treatment in ref. 1, since there exists only one layer inside the transducer, we do not need this distinction.)

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e-mail: michio_ohki@ieee.org

2. Matrix Formulation

In order to avoid the duplication of description in the previous studies and on account of limited space, our attention is focused on the two types of interaction process, point interaction and continuous interaction, mentioned above.

In the case of the point interaction, the interaction process is represented with a matrix operation using a unitary matrix, denoted by P in this study:

$$P = \begin{pmatrix} \sqrt{1-p^2} & p\exp(\mathrm{j}\psi) \\ -p\exp(-\mathrm{j}\psi) & \sqrt{1-p^2} \end{pmatrix} \quad (1)$$

where p and ψ are the amplitude and phase of the coefficient for point interaction, respectively. In the case of one-layer system (N = 1), the two modes vary after the interaction as

$$\begin{pmatrix} \eta'_c \\ \eta'_d \end{pmatrix} = P \begin{pmatrix} \eta_c \\ \eta_d \end{pmatrix} \quad (2)$$

where the prime (') is added for the mode after the interaction, for the convenience of notation. In the case of *N*-layered system, P becomes a *2N*-by-*2N* matrix, in which the interaction of

$$\begin{array}{c} \eta_{c1} \leftrightarrow \eta_{d1} \\ \eta_{c2} \leftrightarrow \eta_{d2} \\ \cdots \\ \eta_{cN} \leftrightarrow \eta_{dN} \end{array}$$

are described, where η_{ci} and η_{di} are the elastic and dielectric modes in the ith layer, respectively.

At the boundary of layer, the reflection and transmission of energy also occur. The reflection and transmission process is expressed with a scattering matrix, which describes the flow of

$$\begin{array}{ll} \eta_{c1} \to \eta_{c1},\ \eta_{c2} \to \eta_{c2} & \text{(reflection)} \\ \eta_{c1} \to \eta_{c2},\ \eta_{c2} \to \eta_{c1} & \text{(transmission)} \\ \eta_{d1} \to \eta_{d1},\ \eta_{d2} \to \eta_{d2} & \text{(reflection)} \\ \eta_{d1} \to \eta_{d2},\ \eta_{d2} \to \eta_{d1} & \text{(transmission)} \end{array}$$

between the 1st and 2nd layers, for example. This scattering matrix (denoted by $\bar{S}$) is associated with matrix P in a cascade manner at the boundary of layer and constructs a new scattering matrix (denoted by S) in the form of

$$S = P\bar{S}P \tag{3}$$

On the other hand, in the case of the continuous interaction, the matrix has the form of the "exponential" of a matrix:

$$\begin{pmatrix} \eta'_c \\ \eta'_d \end{pmatrix} = \exp(x'A) \begin{pmatrix} \eta_c \\ \eta_d \end{pmatrix}, \tag{4}$$

as a solution (integral) of a differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} \eta_c \\ \eta_d \end{pmatrix} = A \begin{pmatrix} \eta_c \\ \eta_d \end{pmatrix} \tag{5}$$

in the case of $N = 1$, where $A\mathrm{d}x$ is a matrix that expresses the interaction process on infinitesimal interaction length $\mathrm{d}x$, and the integral is spatially performed from $x = 0$ to $x = x'$, where x' is an interaction length that is not necessarily equal to an actual spatial length.

Since not only the energy exchange process but also the shift of phase and attenuation of amplitude of modes should be considered in a mixed manner inside the layer, the matrix A for this process is constructed with two types of matrices, one for the shift of phase and attenuation of amplitude, denoted by $\bar{A}$, due to pure propagation process, and the other for the energy exchange process, denoted by U, on which the energy should be conserved; that is, we set the form of

$$A = \bar{A} + U. \tag{6}$$

With regard to the formulation of $\bar{A}$, the integration of the infinitesimal propagation effect from $x = 0$ to $x = x'$ causes the total phase shift and amplitude attenuation, which can be described as

$$\exp(x'\bar{A}) = \begin{pmatrix} \exp(-\mathrm{j}\phi_c) & 0 \\ 0 & \exp(-\mathrm{j}\phi_d) \end{pmatrix}, \tag{7}$$

where ϕ_c and ϕ_d are phase shifts as complex numbers for the elastic mode and the dielectric mode, respectively.

By taking the matrix logarithm of eq. (7),

$$x'\bar{A} = \begin{pmatrix} -\mathrm{j}\phi_c & 0 \\ 0 & -\mathrm{j}\phi_d \end{pmatrix}, \tag{8}$$

is obtained. ϕ_c can be described as

$$\phi_c = \omega T - \mathrm{j}\alpha_c$$

where ω is an angular frequency of the mode as a wave, T and α_c are a propagation time and a loss factor, respectively, when the elastic mode propagates from $x = 0$ to $x = x'$.

Since the dielectric mode is regarded as a "non-propagating" mode, the real part of ϕ_d should be set zero [1]:

$$\phi_d = -\mathrm{j}\alpha_d$$

where α_d is a loss factor for the dielectric mode.

With regard to the formulation of U, the integration of infinitesimal energy exchange between the two modes from $x = 0$ to $x = x'$ can be expressed as

$$\exp(x'U) = \begin{pmatrix} \sqrt{1-u^2} & u\exp(\mathrm{j}\theta) \\ -u\exp(-\mathrm{j}\theta) & \sqrt{1-u^2} \end{pmatrix}, \tag{9}$$

which is also a unitary matrix, where u and θ are the amplitude and phase of the coefficient for continuous interaction, respectively.

By taking the matrix logarithm of eq. (9), we obtain

$$x'U = \log \begin{pmatrix} \sqrt{1-u^2} & u\exp(\mathrm{j}\theta) \\ -u\exp(-\mathrm{j}\theta) & \sqrt{1-u^2} \end{pmatrix}, \tag{10}$$

As a result, for N-layered system, the propagation of the modes is expressed as the following matrix D:

$$D = \begin{pmatrix} \exp(x'_1 A_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \exp(x'_N A_N) \end{pmatrix}, \tag{11}$$

The above matrices S and D are appropriately used for the construction of infinite geometric series (see Table I and Fig. 4 in ref. 2), and the calculation of not only resonance frequencies and resonance intensities but also the spatial distribution of modes becomes possible.

References

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